

Some notes on elliptic equation method

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Abstract

Elliptic equation $(y')^2 = a_0 + a_2y^2 + a_4y^4$ is the foundation of the elliptic function expansion method of finding exact solutions to nonlinear differential equation. In some references, some new form solutions to the elliptic equation have been claimed. In the paper, we discuss its solutions in detail. By detailed computation, we prove that those new form solutions can be derived from a very few known solutions. This means that those new form solutions are just new representations of old solutions. From our discussion, some new identities of the elliptic function can be obtained. In the course of discussion, we give an example of this kind of formula.

Keywords: elliptic equation, elliptic function, exact solution

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1 Introduction

Elliptic equation reads

$$(y')^2 = a_0 + a_2y^2 + a_4y^4 \quad (1)$$

which is utilized to solve various nonlinear differential equations(see Refs.[1-8] and the references therein). For example, Fu et al [1,2] use the solutions of Eq.(1) to give a number of new kinds of solutions to Sinh-Gordon equation and MKdV equation. It is well known that its integral form is as follows

$$\int \frac{dy}{\sqrt{a_0 + a_2y^2 + a_4y^4}} = \pm(\xi - \xi_0). \quad (2)$$

Indeed, all elliptic function solutions of integral (2) can be classified by using direct integral method and complete discrimination system for the third degree polynomial(see, for example, Refs.[9, 10]). All solutions for the integral of a

general elliptic equation $(y')^2 = a_0 + a_1y + a_2y^2 + a_3y^3 + a_4y^4$ can be obtained (for example, see Ref.[11]). In the paper, for brevity, we only consider the case of $a_2^2 - 4a_0a_4 > 0$. Similarly, other cases can be easily dealt with. The following result is well known. For the purpose of completeness and illustration, we list it as follows. If $a_4 > 0$, we rearrange $0, -\frac{1}{b_1}$ and $-\frac{1}{b_2}$, and denote them as $\alpha < \beta < \gamma$. where $b_{1,2} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_0a_4}}{2a_4}$ and $b_1 \neq b_2 \neq 0$. When $\alpha < y^2 < \beta$, we have

$$y = \pm \{ \alpha + (\beta - \alpha) \operatorname{sn}^2(\sqrt{a_4(\gamma - \alpha)}(\xi - \xi_0)), m \}^{\frac{1}{2}}. \quad (3)$$

When $y^2 > \gamma$, we have

$$y = \pm \left\{ \frac{\gamma - \beta \operatorname{sn}^2(\sqrt{a_4(\gamma - \alpha)}(\xi - \xi_0)), m}{\operatorname{cn}^2(\sqrt{a_4(\gamma - \alpha)}(\xi - \xi_0)), m} \right\}^{\frac{1}{2}}, \quad (4)$$

where $m^2 = \frac{\beta - \alpha}{\gamma - \alpha}$.

If $a_4 < 0$, we rearrange $0, \frac{1}{b_1}$ and $\frac{1}{b_2}$, and denote them as $\alpha < \beta < \gamma$. When $\alpha < -w < \beta$, we have

$$y = \pm \{ -\alpha - (\beta - \alpha) \operatorname{sn}^2(-\sqrt{-a_4(\gamma - \alpha)}(\xi - \xi_0)), m \}^{\frac{1}{2}}. \quad (5)$$

When $y^2 < -\gamma$, we have

$$y = \pm \left\{ -\frac{\gamma - \beta \operatorname{sn}^2(\sqrt{-a_4(\gamma - \alpha)}(\xi - \xi_0)), m}{\operatorname{cn}^2(\sqrt{-a_4(\gamma - \alpha)}(\xi - \xi_0)), m} \right\}^{\frac{1}{2}}, \quad (6)$$

where $m^2 = \frac{\beta - \alpha}{\gamma - \alpha}$.

In the present paper, I discuss in detail the solutions of elliptic equation (1) and prove that all of those solutions in Ref.[1] (the results in Ref.[2] can also be dealt with similarly) can be derived from the solutions (3-6), that is, Fu's solutions just give some new representations of solutions to the elliptic equation rather than new solutions. However, we must point out that some new identities of elliptic functions can be obtained by using these new representations. In the course of discussion, we give an example of this kind of formula.

This paper is organized as follows. In section 2, our main results are given. We verify that all solutions of the elliptic equation in reference [1] can be represented by the solutions (3-6). At the same time, a new formula of elliptic function is given. In section 3, we give a summary.

2 Transformations of solutions

For the elliptic equation (1), we verify all of the solutions given in Ref.[1] are the special cases of the solutions (3-6). We first list the addition formulae of elliptic functions as follows:

$$\operatorname{sn}(\xi + \eta) = \frac{\operatorname{sn} \xi \operatorname{cn} \eta \operatorname{dn} \eta + \operatorname{cn} \xi \operatorname{dn} \xi \operatorname{sn} \eta}{1 - m^2 \operatorname{sn}^2 \xi \operatorname{sn}^2 \eta}, \quad (7)$$

$$\operatorname{cn}(\xi + \eta) = \frac{\operatorname{cn} \xi \operatorname{cn} \eta - \operatorname{sn} \xi \operatorname{dn} \xi \operatorname{sn} \eta \operatorname{dn} \eta}{1 - m^2 \operatorname{sn}^2 \xi \operatorname{sn}^2 \eta}, \quad (8)$$

$$\operatorname{dn}(\xi + \eta) = \frac{\operatorname{dn} \xi \operatorname{dn} \eta - m^2 \operatorname{sn} \xi \operatorname{cn} \xi \operatorname{sn} \eta \operatorname{cn} \eta}{1 - m^2 \operatorname{sn}^2 \xi \operatorname{sn}^2 \eta}. \quad (9)$$

For brevity, we only discuss the first ten cases in Ref.[1], other cases can be discussed similarly.

Case 1. $a_0 = 1 - m^2, a_2 = 2m^2 - 1, a_4 = -m^2$. According to Eq.(5), where $\alpha = -1, \beta = 0, \gamma = \frac{1-m^2}{m^2}$, we have

$$y = \pm \operatorname{cn}((\xi - \xi_0), m), \quad (10)$$

which includes the solution in Eq.(16) of Ref.[1].

Case 2. $a_0 = -m^2, a_2 = 2m^2 - 1, a_4 = 1 - m^2$. According to Eq.(4), where $\alpha = -\frac{m^2}{1-m^2}, \beta = 0, \gamma = 1$, we have

$$y = \pm \frac{1}{\operatorname{cn}((\xi - \xi_0), m)}, \quad (11)$$

which includes the solution in Eq.(18) of Ref.[1].

Case 3 and case 4. $a_0 = 1, a_2 = 2 - m^2, a_4 = 1 - m^2$. According to Eq.(4), where $\alpha = -\frac{1}{1-m^2}, \beta = -1, \gamma = 0$, we have

$$y = \pm \frac{\operatorname{sn}((\xi - \xi_0), m)}{\operatorname{cn}((\xi - \xi_0), m)}, \quad (12)$$

which includes the solution in Eq.(20) of Ref.[1].

Case 5. $a_0 = 1, a_2 = 2m^2 - 1, a_4 = m^2(m^2 - 1)$. According to Eq.(5), where $\alpha = -\frac{1}{1-m^2}, \beta = 0, \gamma = \frac{1}{m^2}$, we have

$$y = \pm \frac{1}{\sqrt{1-m^2}} \operatorname{cn}((\xi - \xi_0), m). \quad (13)$$

Since $y(0) = 0$ from the solution in Eq.(24) in Ref.[1], we have $\operatorname{cn}(\xi_0, m) = 0$. Furthermore, we have $\operatorname{sn}^2(\xi_0, m) = 1$ and $\operatorname{dn}^2(\xi_0, m) = 1 - m^2$. According to the addition formula, we have

$$\operatorname{cn}(\xi - \xi_0) = \frac{-\operatorname{sn} \xi \operatorname{dn} \xi \operatorname{sn} \xi_0 \operatorname{dn} \xi_0}{\operatorname{dn}^2 \xi} = \pm \sqrt{1-m^2} \frac{\operatorname{sn} \xi}{\operatorname{dn} \xi}, \quad (14)$$

and hence we have

$$y = \pm \frac{\operatorname{sn} \xi}{\operatorname{dn} \xi}, \quad (15)$$

which includes the solution in Eq.(24) of Ref.[1].

Case 6 and case 7. $a_0 = 1 - m^2, a_2 = 2 - m^2, a_4 = 1$. According to Eq.(4), where $\alpha = -1, \beta = m^2 - 1, \gamma = 0$, we have

$$y = \pm \frac{\sqrt{1-m^2} \operatorname{sn}((\xi - \xi_0), m)}{\operatorname{cn}((\xi - \xi_0), m)}. \quad (16)$$

Since $y(0) = \infty$ from the solution in Eq.(26) in Ref.[1], we have $\text{cn}(\xi_0, m) = 0$. Furthermore, we have $\text{sn}^2(\xi_0, m) = 1$ and $\text{dn}^2(\xi_0, m) = 1 - m^2$. From case 5 we know

$$\text{cn}(\xi - \xi_0) = \frac{\sqrt{1 - m^2} \text{sn} \xi}{\text{dn} \xi}. \quad (17)$$

In addition, according to the addition formula we have

$$\text{sn}(\xi - \xi_0) = \frac{\text{cn} \xi \text{dn} \xi \text{sn} \xi_0}{\text{dn}^2 \xi} = \frac{\text{cn} \xi}{\text{dn} \xi}. \quad (18)$$

Thus we have

$$y = \pm \frac{\text{cn} \xi}{\text{sn} \xi}, \quad (19)$$

which includes the solution in Eq.(26) of Ref.[1].

Case 8. $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, $a_4 = 1$. According to Eq.(4), where $\alpha = -m^2$, $\beta = 0$, $\gamma = 1 - m^2$, we have

$$y = \pm \frac{\sqrt{1 - m^2}}{\text{cn}((\xi - \xi_0), m)}. \quad (20)$$

Since $y(0) = \infty$ from the solution in Eq.(31) in Ref.[1], we have $\text{cn}(\xi_0, m) = 0$. Furthermore we have $\text{sn}^2(\xi_0, m) = 1$ and $\text{dn}^2(\xi_0, m) = 1 - m^2$. From case 5 we know

$$\text{cn}(\xi - \xi_0) = \frac{\sqrt{1 - m^2} \text{sn} \xi}{\text{dn} \xi}. \quad (21)$$

Thus we have

$$y = \pm \frac{\text{dn} \xi}{\text{sn} \xi}, \quad (22)$$

which includes the solution in Eq.(31) of Ref.[1].

Case 9 and case 10. $a_0 = \frac{1-m^2}{4}$, $a_2 = \frac{1+m^2}{2}$, $a_4 = \frac{1-m^2}{4}$. In ref.[1], Fu et al give its two solutions as follows

$$y = \frac{\text{cn}(\xi, m)}{1 \pm \text{sn}(\xi, m)}. \quad (23)$$

On the other hand, according to Eq.(4), where $\alpha = -\frac{1+m}{1-m}$, $\beta = -\frac{1-m}{1+m}$, $\gamma = 0$, we have

$$y = \pm \frac{\sqrt{\frac{1-m}{1+m}} \text{sn}(\frac{1+m}{2}(\xi - \xi_0), k)}{\text{cn}(\frac{1+m}{2}(\xi - \xi_0), k)}, \quad (24)$$

where $k^2 = \frac{4m}{(1+m)^2}$. In order to show the relations of two kinds of solutions obtained by the above two transformations, we give the following lemma.

Lemma: If we take ξ_0 such that

$$\text{sn}(\frac{(1+m)\xi_0}{2}, k) = -\sqrt{\frac{1+m}{2}}, \quad (25)$$

then we have

$$\frac{1 + \operatorname{sn}(\xi, m)}{\operatorname{cn}(\xi, m)} = \sqrt{\frac{1-m}{1+m}} \times \frac{\operatorname{sn}(\frac{1+m}{2}(\xi + \xi_0), k)}{\operatorname{cn}(\frac{1+m}{2}(\xi + \xi_0), k)}, \quad (26)$$

and

$$\frac{1 - \operatorname{sn}(\xi, m)}{\operatorname{cn}(\xi, m)} = \sqrt{\frac{1-m}{1+m}} \times \frac{\operatorname{sn}(-\frac{1+m}{2}(\xi + \xi_0), k)}{\operatorname{cn}(-\frac{1+m}{2}(\xi + \xi_0), k)}. \quad (27)$$

Proof: we consider only the first formula (26). From the solutions (23), we know $y(0) = 1$, and hence we have

$$\int_0^\xi d\xi = \int_1^y \frac{dy}{\sqrt{\frac{1-m^2}{4} + \frac{1+m^2}{2}y^2 + \frac{1-m^2}{4}y^4}}. \quad (28)$$

Take the transformation of variable

$$y = \frac{1 - \sin \varphi}{\cos \varphi}. \quad (29)$$

Then

$$-\xi = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}, \quad (30)$$

and hence we have

$$\frac{1 + \operatorname{sn}(\xi, m)}{\operatorname{cn}(\xi, m)}. \quad (31)$$

On the other hand, we take another transformation of variable

$$y = \sqrt{\frac{1-m}{1+m}} \times \frac{\sin \phi}{\cos \phi}. \quad (32)$$

Then we have

$$\frac{(1+m)\xi}{2} = \int_{\phi_0}^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \int_0^{\phi_0} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad (33)$$

where $k^2 = \frac{4m}{(1+m)^2}$, $\frac{(1+m)\xi_0}{2} = \int_0^{\phi_0} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$ and $\operatorname{sn} \frac{(1+m)\xi_0}{2} = \sin \phi_0 = -\sqrt{\frac{1+m}{2}}$. Thus we have

$$y = \sqrt{\frac{1-m}{1+m}} \times \frac{\operatorname{sn}(\frac{1+m}{2}(\xi + \xi_0), k)}{\operatorname{cn}(\frac{1+m}{2}(\xi + \xi_0), k)}. \quad (34)$$

The proof is completed.

According to Eq.(25) in the above lemma, if we take $\xi_1 = 0$ and ξ_0 satisfies $\operatorname{sn} \frac{(1+m)\xi_0}{2} = \sqrt{\frac{1+m}{2}}$ in solution (24) and use $\frac{\operatorname{cn}(\xi, m)}{1 - \operatorname{sn}(\xi, m)} = \frac{1 + \operatorname{sn}(\xi, m)}{\operatorname{cn}(\xi, m)}$, then we

know solution (23)(take sign "-") and solution (24) (take sign "+") are the same one. Another case is similar.

Eqs.(26) and (27) can be written as a unit form

$$\frac{1 + \operatorname{sn}(\xi - \xi_1, m)}{\operatorname{cn}(\xi - \xi_1, m)} = \sqrt{\frac{1-m}{1+m}} \times \frac{\operatorname{sn}(\frac{1+m}{2}(\xi + \xi_0), k)}{\operatorname{cn}(\frac{1+m}{2}(\xi + \xi_0), k)}, \quad (35)$$

where $\operatorname{sn}(\frac{(1+m)(\xi_0 + \xi_1)}{2}, k) = -\sqrt{\frac{1+m}{2}}$. This is just a new identity of elliptic functions.

It is easy to see that when $m \rightarrow 0$, we have $\operatorname{sn} \rightarrow \sin$, $\operatorname{cn} \rightarrow \cos$. Therefore, if $\sin \frac{\xi_0}{2} = -\sqrt{\frac{1}{2}}$, e.g., $\xi_0 = -\frac{\pi}{2}$, we obtain from Eqs.(25-27),

$$\frac{1 + \sin \xi}{\cos \xi} = \frac{\sin(\frac{1}{2}(\xi + \xi_0))}{\cos(\frac{1}{2}(\xi + \xi_0))}, \quad (36)$$

and

$$\frac{1 - \sin \xi}{\cos \xi} = \frac{\sin(-\frac{1}{2}(\xi - \xi_0))}{\cos(-\frac{1}{2}(\xi - \xi_0))}. \quad (37)$$

In fact, these above two equations can be verified easily.

Remark 1. Indeed, we only use two solutions (4) and (5) to verify the first ten cases in Ref.[1].

Remark 2. Similarly, we can also verify that other twelve solutions given in Ref.[2] for elliptic equation can be represented by the solutions (3-6).

3 Conclusions

We verify that all solutions of elliptic equation obtained in Ref.[1] can be represented by the solutions (3-6). This fact means that some new form solutions are only new representations of elliptic functions. Therefore, some solutions claimed in the references [1,2] are not novel. However, by using those solutions, some interesting new formulae of elliptic functions can be obtained.

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